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Potsdam, New York 13676

SYNTHESIS OF STIFFENED SHELLS OF REVOLUTION

FINAL REPORT

(covering the period June 1969-May 1974)

NASA GRANT NGL 33-007-075



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#### SUMMARY

The work done under this grant can be broken into three phases. A description of these phases is given in Table 1. Additional comment on these phases follows.

Phase I (1969-1971) Design of uniform shells (mainly cones) using a membrane prebuckling analysis, the Langley developed SALORS program for general buckling, the Fiacco McCormick<sup>2</sup> and Powell<sup>3</sup> Penalty functions, and Davidon Fletcher Powell<sup>4</sup> and Marquardt's<sup>5</sup> methods for function minimization. Seven design variables were considered. All derivatives exact except general buckling load derivative. This work was an extension of Morrow and Schmit's<sup>6</sup> work. Work was published in Journal of Spacecraft and Rockets.<sup>7</sup>

Phase II (1971-1973) Design of completely nonuniform shells (cones, spheres, toriodal segments) using a linear bending prebuckling analysis. The design problem was formulated and solved as a control theory problem. A first order differential equation formulation was used. General buckling was not included in the design process but was checked after a design was obtained for local yield and buckling constraints. A total of 245 design variables were used. This work is described in the Ph.D. thesis of D.K. Majumder,<sup>8</sup> a graduate student supported by the grant.

Also during this period, difficulties with the SALORS problem were encountered and a new program, named SORAN (Shell of Revolution Analysis) and based on Cohen's<sup>9</sup> computer approach, was developed.

Phase III (1973-1974) The design process of Phase II was revised to reduce the number of design variables to about 30 by considering piecewise uniform designs. Also, the SORAN program was incorporated into the design scheme so that general buckling could once again be considered in the design process. A perturbation formula based on the SORAN program was derived and this allows exact derivatives of the general buckling load to be computed with very little additional computer time. A paper<sup>10</sup> which presents this formula has been accepted for publication in the AIAA Journal.

In order to reduce the computer time required, a mixed interior-exterior penalty function is used which allows a reduction in the number of times the

Table 1. Outline of Work Accomplished Under NASA Grant NGL-33-007-075.

Phase/Dates	Description of Work	Publication or Thesis	Significant Results
I/1969-1971	Design of uniform stiffened shells, membrane prebuckling state, SALORS Buckling program. Exact derivatives except for general buckling. Seven design variables. Two penalty functions and two unconstrained minimization techniques tried. Results obtained for cones and cones with slight meridional curvature.	"Synthesis of Stiffened Conical Shells," <u>Journal of Spacecraft and Rockets</u> , Vol. 9, No. 3, March 1972, pp. 189-195.	Direct extension of work of Morrow and Schmit as reported in NASA CR 1217, December 1968.
II/1971-1973	Design of completely nonuniform stiffened shells, linear bending prebuckling state, general buckling tacked on after termination of design process. Exact derivatives of all constraints obtained through a control theory approach. A total of 245 design variables were considered. Results for cones, spheres, and toriodal segments obtained. Also a new general buckling program (SORAN) based on Cohen's shell formulation was developed to allow general buckling to be brought back into the design process in the next phase.	Ph.D. Thesis by D.K. Majumder submitted to Clarkson College of Technology, September 1973.	1.) First attempt at automated preliminary design of general shells with nonuniform wall thickness and nonuniform stiffeners. Direct extension of work of Cohen as reported in NASA CR 1424, August 1969. Limitation is exclusion of general backing constraint. Also, a piecewise uniform configuration would be more practical. 2.) SORAN program developed-provides nucleus for phase III of work.
III/1973-1974	Design of piecewise uniform shells. Discrete rings treated as prescribed parameters. Linear bending prebuckling analysis and SORAN buckling analysis included in	1.) "Reanalysis Information for Eigenvalues derived from a differential	1.) Practical approach to preliminary design of non-uniform shells because of piecewise uniformity of shell $\omega$

(Continued next page)

Table 1 (Continued)

	<p>design formulation. Exact derivatives of all constraints included (for the general buckling load this required the derivation of a new formula for eigenvalue perturbations.). A total of 30 design variables are considered. A mixed interior-exterior penalty function is used in conjunction with the Fletcher-Reeves method.</p>	<p>Equation Analysis Formulation," accepted for publication in the <u>AIAA Journal</u>.  2.) "Optimal Stiffened Shells of Revolution," to be submitted for possible publication (see Appendix this report).</p>	<p>which should allow easy fabrication.  2.) Self Adjoint formulation of SORAN buckling program allows efficient calculation of derivatives of general buckling load.  3.) Perturbation formula for buckling loads of a general shell of revolution derived.  4.) The Phase III design capability is probably the most general preliminary static design capability presently available for shells of revolution.</p>
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general buckling load must be computed. Also, this latest approach allows discrete rings to be included in the design process as prescribed parameters. A paper describing this latest design approach has been written (see Appendix) and will be submitted for publication shortly. The capability presented in this paper is felt by the principal investigator to be the most general preliminary shell of revolution static design capability presently available.

#### Detailed Description of Work Done

A total of ten Status Reports have been submitted during the course of the grant. These give the detailed description of the work done. The work done during the final report period (Dec. 1973 to May 1974) is contained in the Appendix to this Final Report in the form of a paper to be submitted for publication. This paper and the others noted in Table 1 will complete the dissemination in the open literature of the work described in the various Status Reports.

#### General Discussion

##### Goal of Research - Limitations - Art of Design

The goal of this research was to attempt the automation of the design process for general stiffened shells of revolution. In the context of preliminary design of parts of a complex shell structure such as a missile (S4-B etc.), this goal has been accomplished. Two qualifying words (preliminary and parts) need further clarification.

By "preliminary" design a preliminary sizing of the elements of the shell by means of relatively crude design formulas is meant. For a stiffened shell, this usually means computing the general buckling load with "smeared" stiffeners (the equivalent shell is an orthotropic monocoque shell) and treating the failure modes of the stiffeners themselves and the shell wall between stiffeners as independent (or uncoupled).

By "parts" of a shell a distinction is made between the shell structure which generally will be fabricated of several shells joined together by discrete stiffening rings and the "part" of the shell structure between

these discrete rings.

In the design of any shell structure composed of several parts which are connected by stiffening rings, these rings must of course also be designed. At the present state of the art, it is felt that the design of these rings is best left to the experienced designer. With the preliminary designs for the parts of the shell obtained from the capability developed under this Grant, and an advanced computer capability (such as NASTRAN or the program developed by Bushnell<sup>11</sup> for handling branched shells), the experienced designer can reliably size the stiffening ring (or rings) and make whatever modifications are necessary in the preliminary designs of the shell parts to arrive at a final design.

There are two reasons why it is felt that the final design including connections is best done manually at present.

1.) There is quite a bit of art involved in the design process. It is one thing to automate the design of parts of a structure and quite another to automate the design of a structure as a whole. As an example of the kind of thing involved here, consider the approach to design of the stiffening ring used in Cohen's<sup>12</sup> NASA CR 1424 (Aug. 1969). Here, the stiffening ring at the base of the shell is sized to suppress the  $n = 2$  shell buckling mode. This allows the design of the shell to be uncoupled from the design of the stiffening ring. While the entire shell (shell plus end rings) could be designed by an automated process, it seems clear that Cohen's approach is the superior one.

2.) If all the art involved in design were able to be quantified into a computer automated design approach, the only way a final design could be obtained directly would be by incorporating such programs as Bushnell's branched shell analysis into the design process. Considering the core and run time necessary for such programs, this seems prohibitively expensive at the present time.

### Analysis Methods for Design

It has been noted by Storaasli and Sobieszczanski<sup>13</sup> that a need exists for analysis techniques which are developed with an eye on their use in design. The basic idea here is to develop analysis techniques which do not require complete re-solution when small changes are made in parameters which govern the results obtained from the analysis. Then, using, for instance, a Taylor series expansion in the design variables about the initial design point, analyses for neighboring points can be determined without re-solving the problem if the derivatives are available from the first solution. In buckling problems, symmetry and self-adjointness are properties of the formulation which are central to efficient re-analysis, since in this case the required derivatives can be calculated without re-analyzing the problem. Since the SALORS program is not a symmetric-formulation, it is difficult to use in a redesign environment. This prompted the development at Clarkson of the SORAN program which has the self-adjoint property. Basically, the difficulty with nonsymmetric or nonself adjoint formulations is that either the adjoint eigenvalue problem must be solved or the derivatives of the eigenvectors must be found in order to get the derivatives of the eigenvalues. If the adjoint problem is solved, generally its solution is just as costly as the solution to the original eigenvalue problem, thereby greatly degrading the efficiency of the design program.

The design programs developed during Phase III of this Grant (Table 1) are based on self-adjoint prebuckling and buckling analyses and this property is exploited to create an efficient redesign process. The prebuckling and buckling formulations are both formulated as sets of first order ordinary differential equations. These equations are integrated by the fourth order Runge Kutta method and the integration method of Anderson, et al.<sup>1</sup> For the design of piecewise uniform shells, this first order formulation has the following advantages over the second order approach of SALORS:

- 1.) The shell must be segmented to provide for an accurate solution. This segmentation allows for an obvious piecewise uniform shell. The SALORS approach requires no segmentation and thus an artificial segmentation must be introduced to provide for piecewise uniformity of the shell.
- 2.) The first order formulation requires no derivatives of design variables

with respect to meridional position whereas SALORS does. At points of segmentation these derivatives do not exist. Hence, in SALORS, some special steps must be taken to compute these derivatives on either side of the segmentation point.

#### Optimality Criterion for Design

The design method developed under this grant uses the versatile math programming techniques and thus is guaranteed to converge to at least a local minimum. During the last few months of the grant, an optimality criterion approach based on a variable energy method,<sup>14</sup> which was developed at Clarkson for use in frame and curved beam design, has been applied to shell design. Excellent results were obtained for a monocoque cone. Considering 6 segments and stress and general buckling constraints, a minimum weight design was obtained with only 8 reanalyses and no derivative calculations. This can be contrasted with about 90 reanalyses (about 30 of which also require derivative calculations) which would be required for the math programming method.

At present the energy ratio method is limited to one design variable per segment. That is why a monocoque shell was studied. For the stiffened shell, there are 5 design variables per segment. For the energy approach to be applied to these shells either 4 relationships must be introduced to reduce the 5 design variables to 1 or some more general energy approach must be taken. Some preliminary studies considering the energy in the wall as a whole as opposed to the separate energies of the parts of the wall (skin, rings, stringers) have been made but no general results have been obtained and no conclusions can be made at present about this approach. The great potential reduction in reanalyses (90 to 8) required for an optimal design should provide the impetus to explore further an energy based optimality criterion method for stiffened shell design.



### Directions for Further Research

- 1.) The prebuckling and buckling analyses have been formulated and utilized within the design process in a very efficient manner. The math programming method used could be improved. The method of Schmit and Farshi<sup>15</sup> using "hyper-circles" appears promising in this respect. Alternately, an energy based optimality criterion method for multiple design variables per segment should be developed. Preliminary results (as noted in the preceding section of this report) indicate that a method of this kind would greatly reduce the number of reanalyses required for an optimal design.
- 2.) If an optimality criterion method proves feasible for this problem, the number of reanalyses required for an optimal design can be expected to be reduced by a factor of about 10. With this kind of reduction, it begins to become feasible to consider incorporating a shell analysis using discrete rather than smeared rings into the design process. This would give a more accurate assessment of local ring buckling (stringers would still be "smeared"), skin yielding and skin buckling. The resulting designs would still be preliminary designs because the connections of the shell (or shell part) to its environment would very likely still be considered manually by the designer.

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#### Appendix

Contains a paper which will be submitted for possible publication.  
This paper contains the work done during the last report period,  
Dec. 1973 - May 1974.

# Optimal Stiffened Shells of Revolution<sup>1</sup>

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## Summary

A method to produce piecewise uniform optimal stiffened shells of revolution is presented. The method uses a first order differential equation formulation for the shell prebuckling and buckling analyses and the necessary conditions for optimality are derived by a variational approach. A variety of local yielding and buckling constraints and the general buckling constraint are included in the design process. The local constraints are treated by means of an interior penalty function and the general buckling load is treated by means of an exterior penalty function. This allows the general buckling constraint to be included in the design process only when it is violated. The self adjoint nature of the prebuckling and buckling formulations is used to reduce the computational effort. Results for four conical shells and one spherical shell are given.

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### Introduction

Several papers have dealt with the problem of optimal shell design. Among these are papers by Salama and Ross<sup>(1)</sup>, Cohen<sup>(2)</sup> and Thornton<sup>(3)</sup>. Salama and Ross<sup>(1)</sup> consider nonuniform unstiffened shells for stress and natural frequency constraints. They use a finite element computer program for the analysis and couple this with two alternative mathematical programming techniques. Only one load condition is considered, but the generality of the approach is such that multiple load conditions could easily be considered. Cohen<sup>(2)</sup> considered stiffened conical shells for local and general buckling constraints. One load condition, uniform external pressure is considered. A membrane prebuckling state is used for all local constraints. An empirical correlation formula is used to include the general buckling load in the design process. A simultaneous failure mode design technique is used. The wall thickness is uniform and nonuniform ring size and spacing are included. Stringers are not designed. Thornton<sup>(3)</sup> has treated conical ring and stringer stiffened shells. Local stress and buckling constraints and general buckling are included, the latter by means of a general shell of revolution buckling analysis, the SALORS<sup>(4)</sup> program. The local constraints are based on a membrane prebuckling state. The wall thickness is uniform, as is the ring and stringer size and spacing. The design process uses a mathematical programming technique. Results for one load condition, uniform external pressure, are presented, but the generality of the approach permits the inclusion of multiple load conditions.

This paper sketches an approach to shell design which is felt to be an extension of the approaches used in the above papers, because: 1. linear bending shell theory is used for the prebuckling analysis and all local constraints, 2. the shell wall thickness, the stiffener size parameters, and the stiffener spacings, are treated as piecewise uniform functions of meridional position, 3. the general buckling load is used as a constraint only when it plays an active role in the design, and 4. results for conical and spherical shells have been obtained.

### Structural Analysis

The prebuckling behavior of the shell is assumed axisymmetric. In this case, the prebuckling equations can be written as six first-order ordinary differential equations as

$$\begin{aligned} ry' + ay - bz &= p \\ rz' - cy + dz &= 0 \end{aligned} \quad (1)$$

where the prime denotes differentiation with respect to meridional position,  $s$ ,  $y$  is a vector of meridional forces,  $y^T = [P, Q, M, \dots]$ ,  $z$  is a vector of meridional displacements,  $z^T = [\xi, \eta, \chi]$ ,  $r$  defines meridional position,  $p$  is a vector of loads per unit area,  $p^T = [X_1, X_2, L_1]$ , and  $a$ ,  $b$ ,  $c$ , and  $d$  are matrices which satisfy the self adjointness conditions

$$a+d^T = r'I, \quad b = b^T, \quad c = c^T \quad (2)$$

The notations for  $y$ ,  $z$ , and  $p$  are from Ref. 5, and the coefficient matrices  $a$ ,  $b$ ,  $c$ , and  $d$  can be obtained indirectly from Ref. 5.

The general buckling analysis can be formulated as eight first order ordinary differential equations as

$$\begin{aligned} rY' + AY - BZ - \lambda_e F &= -\mu\bar{\alpha}Y - \mu\bar{\beta}Z + \mu F \\ rZ' - CY + DZ &= -\mu\bar{\delta}Z \end{aligned} \quad (3)$$

where the notation of Eqs. 6 of Ref. 6 is followed except that upper case  $Y$  and  $Z$  are used here to distinguish the eigenvector of meridional forces and displacements from the prebuckling forces and displacements. The coefficient matrices  $\bar{\alpha}$  and  $\bar{\beta}$  can be found in Ref. 6 and the matrices  $A$ ,  $B$ , and  $C$  can be found in Ref. 7 (denoted  $a$ ,  $b$ ,  $c$  there). Also,  $F$  is a vector of perturbation forces,  $\lambda_e$  is a reference buckling load and  $\mu$  is the buckling load. The matrices  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $\bar{\alpha}$ ,  $\bar{\beta}$ , and  $\bar{\delta}$  satisfy the self adjointness conditions as

$$A + D^T = r'I, B = B^T, C = C^T, \bar{\alpha} + \bar{\delta}^T = 0, \bar{\beta} = \bar{\beta}^T \quad (4)$$

Eqs. 1 and Eqs. 3 are solved by the integration method of Appendix A of Ref. 4. In order to provide for an accurate solution to these equations, the shell must usually be segmented meridionally. Also, if concentrated loadings or discrete rings are present, the shell must be segmented to accommodate these loads or rings. The conditions that must be satisfied by the prebuckling equations at points of segmentation are

$$r\Delta y - kz = L, s = s_i \quad (5)$$

where  $s_i$  denotes the points of meridional segmentation,  $L$  is a matrix of applied loads and  $k$  is the discrete ring stiffness matrix which can be found in Ref. 7. The notation  $\Delta y$  is as defined by Eq. 11 of Ref. 7. The matrix  $k$  is symmetric and this symmetry is necessary for a self adjoint formulation. Conditions at the top and bottom of the shell are a special case of Eq. 5 and classical supports can also be considered by replacing Eq. 5 by the appropriate classical condition, i.e., hinge, roller, fixed, etc. The only requirement on the end conditions is that they result in a self adjoint formulation.

At points of segmentation, the buckling analysis must satisfy the conditions

$$r\Delta Y - KZ = -\mu KZ \quad s = s_i \quad (6)$$

where  $K$  is given by Ref. 7 and  $\mu$  can be found in Ref. 6. Both  $K$  and  $\mu$  are symmetric (which property is necessary for self adjointness). As in the case of the prebuckling analysis, classical conditions at the ends can also be considered but here again it is important that these classical conditions provide a self adjoint formulation. The reason for this will be seen later.

When the prebuckling and buckling equations have been solved, the response of a particular shell design is available and the adequacy of this design can be determined. The adequacy of a design is judged with respect



to constraints on the response of the design. If the response does not violate the constraints, the design is acceptable, otherwise not. Defining the "response ratio"  $\phi_i = (\text{stress in wall})/(\text{yield stress})$ , if  $\phi_1 \leq 1$ , the design is acceptable with respect to wall stress. Other constraints can be put in the same form and if  $\phi_i \leq 1$  for all  $i$  the design is acceptable. The constraints considered in this paper are 1) skin yield, 2) stringer yield, 3) ring yield, 4) skin buckling, 5) circumferential panel buckling, 6) meridional panel buckling, 7) normal displacement, 8) meridional rotation ( $\chi$ ), and 9) general buckling. Minimum gage and fabricational constraints are also included. Discussions on constraints 1, 2, 3, 4, 5, and 9 are contained in Ref. 3. Normal displacement (6) and meridional rotation (7) are self explanatory. Meridional panel buckling (8) considers the situation that would exist if the shell were to be stiffened with stringers but no rings. The meridional panel is a panel imagined to exist at each meridional position which is of infinite length in the meridional direction and has a width equal to the stringer spacing at the point under consideration. This constraint complements the circumferential panel buckling constraint (called panel buckling in Ref. 3) because circumferential panel buckling handles the case where only rings stiffen the shell.

Local buckling of the stiffeners is not considered as a constraint. Rather, each stiffener is proportioned to force stiffener yielding and buckling to occur simultaneously, thus reducing the number of design variables.

### Structural Design

Once a particular design has been judged acceptable (it is assumed that at least one such design can be found) the problem becomes one of determining whether or not this particular design is the best obtainable. The best (or optimal) design will here be considered to be the one which weighs the least. The weight of the shell is given by  $W = \sum_{k=1}^m \int_{\ell_k} W_k ds$  where  $m$  is the

number of segments into which the shell is partitioned and

$$W_k = 2\pi\gamma_W t_{W_K} r + 2\pi\gamma_R \frac{A_{R_K}}{S_{R_K}} r + 2\pi\gamma_S r_K \frac{A_{S_K}}{S_{S_K}} \quad (7)$$

In Eq. 7,  $\gamma$  = weight density,  $t$  = wall thickness,  $A$  = stiffener area,  $S$  = stiffener spacing, subscript  $W$  refers to the wall, subscript  $R$  refers to the rings and subscript  $S$  refers to the stringers. Also,  $l_k$  is the length of the  $k$ th segment and  $r_k$  is the radius of the shell at the beginning of the  $k$ th segment. The stringer spacing  $S_{S_k}$  is defined at the beginning of the  $k$ th segment. The areas  $A_R$  and  $A_S$  are functions of the shape of the rings and stringers. For circular rings of thickness  $t_R$  and diameter  $D_R$ ,  $A_R = \pi t_R D_R$ . For Zee rings of flange width/web height ratio of 2/5, thickness  $t_R$  and web height  $D_R$ ,  $A_R = 1.8 t_R D_R$ . Thus in general

$$A = A(t, D) \quad (8)$$

The quantities  $t_w$ ,  $t_s$ ,  $t_R$ ,  $D_s$ ,  $D_R$ ,  $S_s$ , and  $S_R$ , which in general are different in each segment  $k$  of the shell, are called the design variables. In order to reduce the number of design variables, the stiffeners are assumed to yield and buckle locally simultaneously. Thus, for circular stiffeners, the buckling stress<sup>8</sup>  $\sigma_B$  is equated to the yield stress  $\sigma_y$ ,

$$\sigma_B = .4E t/D = \sigma_y \quad (9)$$

which yields

$$t = (\sigma_y / .4E) D \quad (10)$$

For Zee stiffeners<sup>2</sup>

$$\sigma_B = 3.35E \left(\frac{t}{D}\right)^2 = \sigma_y \quad (11)$$

which yields

$$t = (\sigma_y / 3.35E)^{1/2} D \quad (12)$$

and other formulas can be derived for other stiffener shapes. In Eqs. 9-12,  $E$  is Young's modulus.

With the stiffener thicknesses eliminated through Eqs. 10 and 12, the design variables reduce to  $t_w$ ,  $D_s$ ,  $D_R$ ,  $S_s$  and  $S_R$ . These five are collected into a vector  $u$  in the above order.

A concise statement of the design problem can now be given as; find  $u = u^*$  such that  $W(u^*) \rightarrow \min$  while  $\phi_i(u^*) \leq 1$ , all  $i$ . This design problem can be formulated in the following way. Introduce a new notation for the constraints as  $f_i = 1 - \phi_i$  so that an acceptable design is one for which  $f_i \geq 0$ . Let all constraints except the general buckling constraint have indices which are members of a set  $I$  while the general buckling constraint has an index which is the (sole) member of a set  $O$ . Now define a function  $P_k$  where

$$P_k = W_k + \rho \sum_{i \in I} \frac{1}{f_{k_i}} + \frac{1}{\rho} \sum_{i \in O} H_i(f_{k_i}) f_{k_i}^2 \quad (13)$$

In Eq. 13,  $\rho$  is a parameter which approaches zero sequentially and  $H$  is the Heaviside function,  $H_i = 0$  when  $f_{k_i} \geq 0$  and  $H_i = 1$  when  $f_{k_i} < 0$ . Combining the function  $P_k$  with the prebuckling equations using Lagrange multipliers  $\bar{y}$  and  $\bar{z}$  gives

$$J = \sum_k \int_{\ell_k} [P_k + \bar{z}^T (ry' + ay - bz - p) - \bar{y}^T (rz' - cy + dz)] ds \quad (14)$$

The design problem can be stated in terms of  $J$  as; find  $u = u^*$  such that  $J(u^*, y, z, \bar{y}, \bar{z}, \mu, \rho) \rightarrow \min$  for a monotone decreasing sequence of values of  $\rho$ . This is effectively a continuous form of the well known Fiacco-McCormick<sup>(9)</sup> penalty function method. The necessary conditions for a stationary value of  $J$  in Eq. 14 are obtained by taking the variations of  $J$  with respect to  $u$ ,  $y$ ,  $z$ ,  $\bar{y}$ , and  $\bar{z}$  and setting  $\delta J = 0$ . The variations in  $\bar{y}$  and  $\bar{z}$  result in the reappearance of the prebuckling equations, the variations in  $y$  and  $z$  give the adjoint equations, and the variation in  $u$  gives the gradient of  $J$  which

must vanish for  $\delta J = 0$ . Thus, the necessary conditions are, for  $\delta \bar{z}$  and  $\delta \bar{y}$

$$\begin{aligned} ry' + ay - bz &= p \\ rz' - cy + dz &= 0 \end{aligned} \quad (15)$$

and for  $\delta y$  and  $\delta z$

$$\begin{aligned} r\bar{y}' + a\bar{y} - b\bar{z} &= - \left( \frac{\partial P_k}{\partial z} \right)^T \\ r\bar{z}' - c\bar{y} + d\bar{z} &= - \left( \frac{\partial P_k}{\partial y} \right)^T \end{aligned} \quad (16)$$

while for  $\delta u$

$$\begin{aligned} G_k &= \int_{\ell_k} \left\{ \frac{\partial P_k}{\partial u_k} + \frac{\partial P_k}{\partial \mu} \frac{\partial \mu}{\partial u_k} + \bar{z}^T \frac{\partial a}{\partial u_k} y - \bar{z}^T \frac{\partial b}{\partial u_k} k \right. \\ &\quad \left. - \bar{y}^T \frac{\partial c}{\partial u_k} y + \bar{y}^T \frac{\partial d}{\partial u_k} z \right\} ds = 0 \end{aligned} \quad (17)$$

Eq. 17 is a shorthand notation. Writing Eq. 17 out for the  $j$ th element of  $u_k$  gives

$$G_{jk} = \int_{\ell_k} \left\{ \frac{\partial P_k}{\partial u_{jk}} + \frac{\partial P_k}{\partial \mu} \frac{\partial \mu}{\partial u_{jk}} + \sum_p \sum_q \bar{z}_p \frac{\partial a_{pq}}{\partial u_{jk}} y_q + \dots \right\} ds \quad (18)$$

Regarding Eqs. 15 and 16 it will be noted that they are formally self adjoint (they are not completely self adjoint because the right hand sides differ). Since the boundary and intermediate conditions for Eqs. 15 (the prebuckling equations) produce a self adjoint formulation and are unaffected by the terms on the right hand side of Eqs. 16, the boundary and intermediate conditions are the same for Eqs. 15 and Eqs. 16. This is important because it allows the 3 complementary solutions to Eqs. 15 obtained by the method of Appendix A of Ref. 4 to be used as the 3 complementary solutions for Eqs. 16. Thus, only one new particular solution must be obtained to solve Eqs. 16 once Eqs. 15 have been solved.

It will be noted that no explicit use of the buckling solution given by Eqs. 3 and 5 has appeared in this section of the paper except for the appearance of  $\mu$  in the constraint  $f_i$ ,  $i \in O$  and the derivatives  $\partial\mu/\partial u_k$  in Eq. 17 (and Eq. 18). Eqs. 3 and 5 must be solved to provide  $\mu$  for  $f_i$ ,  $i \in O$  to allow evaluation of  $P_k$  in Eq. 13. The terms  $\partial\mu/\partial u_k$  are obtained from Ref. 10. For the  $j$ th design variable and for the assumed critical  $p$ th buckling mode,

$$\begin{aligned} \frac{\partial\mu_p}{\partial u_{jk}} = & \int_{\ell_k} \left\{ 2Z_p^T \frac{\partial A}{\partial u_{jk}} Y_p - Z_p^T \frac{\partial B}{\partial u_{jk}} Z_p + Y_p^T \frac{\partial C}{\partial u_{jk}} Y_p \right\} ds + \mu_p \int_{\ell_k} \left\{ 2Z_p^T \frac{\partial \bar{\alpha}}{\partial u_{jk}} Y_p \right. \\ & \left. + Z_p^T \frac{\partial \bar{\beta}}{\partial u_{jk}} Z_p \right\} ds - \sum_{s_i} Z_p^T \frac{\partial K}{\partial u_{jk}} - \mu_p \sum_{s_i} Z_p^T \frac{\partial \kappa}{\partial u_{jk}} Z_p \Big/ \int \\ & \left\{ -2Z_p^T \bar{\alpha} Y_p - Z_p^T \bar{\beta} Z_p + F_p^T Z_p \right\} ds + \sum_{s_i} Z_p^T \kappa Z_p \Big\} \end{aligned} \quad (19)$$

The derivation of this formula depends on the self adjointness of the buckling formulation.

All of the derivatives in Eq. 17 (Eq. 18) can now be computed in closed form. Eq. 17 is used as the gradient for the Fletcher-Reeves method<sup>(11)</sup> that is used to solve the sequence of unconstrained minimization problems which result from the introduction of the penalty function. The Fletcher-Reeves method was used in preference to the Davidon Fletcher Powell method because it uses less core and core was at a premium in the computer used (IBM 360/44). Of course, peripheral storage could be used and would most likely result in improved run times.

One final point to be noted about the derivatives given by Eq. 19 is that the matrices  $\bar{\alpha}$  and  $\bar{\beta}$  are functions of  $u$  directly and also implicitly through dependence on the prebuckling solution  $y$  and  $z$ . This latter dependence is neglected in computing  $\partial\mu/\partial u$  with no noticeable effect on the results.

### Results

The shell design method outlined in the foregoing sections has been used to obtain designs for a number of stiffened shells. All shells studied were broken into six segments and 11 finite difference stations were used per segment. The size of the computer used (IBM 360/44) dictated these numbers. Also, because this computer carries only about six significant figures and truncates rather than rounds, double precision was required to obtain reliable results for the general buckling calculation.

The manner in which the penalty function given by Eq. 13 was used varied depending on the problem. In some cases, the general buckling constraint was completely ignored until convergence was obtained for the constraints depending on the prebuckling analysis. Then if general buckling was violated, the program was run further until this violation was eliminated. In other cases, general buckling was checked in the beginning and at each change in the  $\rho$  parameter. It is not possible to give one way which is best for all cases. The designer is free to choose his own variation in approach.

Each shell studied was broken into six segments as mentioned before. Since there are five design variables per segment each shell design involves a total of 30 design variables (25 when stringers are prescribed to be continuous from top to bottom of the shell). Considering the complexity of the analysis for this problem, 30 is a sizeable number of design variables. It will be recalled that the Fletcher Reeves (as with the Fletcher Powell) method requires a one dimensional search to be performed for each search direction computed and that for a convex function of  $n$  variables, in general  $n$  one dimensional searches will be required for convergence. In the present problem  $n = 30$  and the function to be minimized is not necessarily convex which means that generally more than 30 one dimensional searches will be required for convergence. Assuming for the sake of argument that 30 one dimensional searches lead to convergence, that each one dimensional search requires approximately 3 function evaluations, and that convergence is required for a sequence of at least 5 values of  $\rho$ , it can be seen that something on the order of 450 function evaluations will be required to complete a design. It should now be clear why 30 is considered to be a sizeable number of design

variables for a problem of this complexity.

Results for four cone designs and one sphere design are presented here. Only one load condition is considered in each case. Multiple load conditions have been treated in cases not presented here and cause no difficulty except for a great increase in computer time required for a solution.

Case I. This is a  $70^\circ$  cone with the geometry and boundary conditions of the cone considered in Ref. 10. The loading is a uniform external pressure of 3.75 psi and the material is Mg with  $\sigma_y = 16000$  psi,  $E = 5.8 \times 10^6$  psi and density  $\gamma = .066$  lbs/in<sup>3</sup>. The cone is stiffened with exterior Z stringers and interior O rings. The stringers are continuous. Table 1 gives the design variables. The shell weighs 51.52 lbs. The constraints which control the design (active constraints) are ring yield (.83), skin yield (.86), stringer yield (.92), and skin buckling (.99). The numbers in parentheses are the response ratios. The general buckling load is 4.67 psi and the critical circumferential mode is 5. The minimum gage for the stiffeners is .01 inches and for the wall .016 inch. Table 1 shows the rings to be minimum gage constrained in all segments and the wall thickness to be approaching minimum gage in segment 6. The design process started from a uniform design with  $t_w = .027$ ,  $D_s = 1.17$ ,  $D_R = 1.02$ ,  $S_s = 1.26$ , and  $S_R = 1.07$ . This design weighed 76.28 lbs. The number of function evaluations required to determine the final optimal 51.52 lb design was 368. Note that a new search direction must be computed approximately every 3rd to 4th function evaluation. The search direction requires that derivatives of the objective function and constraints be computed. These derivatives are easily computed and require only 1 particular solution for Eqs. 16 and the evaluation of Eq. 19 for each design variable.

Case II. This also is a  $70^\circ$  Mg cone with geometry and boundary conditions identical to those of Case I. The cone is stiffened in interior Z stringers and rings. The stringers are continuous. Table 2 gives the design variables. The shell weighs 53.83 lbs, a small increase over the weight for Case I. The constraints which control the design are stringer yield (.99), skin yield (.98),

and skin buckling (.99). The general buckling load is 5.21 psi in mode number 6. No minimum gage constraints are active. The design process started with the same uniform design as that used for Case I (which in this case weighs 93.0 lbs because of the use of Z rings). Convergence to the optimal 53.83 lb design required 250 function evaluations.

Case III. This again is a 70° cone with geometry and boundary conditions as in Case I and II. The stiffeners are internal Z stringers and rings, and the stringers are here taken to be discontinuous (stringer spacing at the beginning of any segment is independent of the stringer spacing at the beginning of each other segment - this gives a total of 30 design variables). The design process started with the same design variable values as in Cases I and II but because of the stiffener change, this design now weighs 107.14 lbs. Table 3 gives the design variables of the final optimal design which weighs 57.58 lbs. This weight represents a 7% increase over the weight of Case II. The constraints which control the design are stringer yield (.97), wall buckling (.99), and general buckling (.97). The general buckling load is 3.85 psi with mode 6. No minimum gage constraints are active. Convergence to the optimal design required 454 function evaluations.

It was noted above that this design weighs 7% more than the design of Case II. Since the stringers are free to take the best spacing value in each segment, it was expected that the weight for this case would be less than the weight for Case II rather than more. It is expected that the use of more stringent convergence criteria would force the weight of this case to less than that of Case II.

Case IV. As in Cases I, II, and III, this shell is a 70° Mg cone again with the geometry and boundary conditions of Ref. 11. The stiffeners are the same as those of Case I. The loading is a line load of 147.66 lb/in applied at the center of the meridian (this is the segmentation point between segments 3 and 4). The load of 147.66 lb/in was chosen so that the total load applied to the shell is the same as the total for the 3.75 psi uniform load. The design process started with a uniform design with  $t_w = 0.8$ ,  $D_s = 1.75$ ,  $D_s = 1.75$ ,  $S_s = 2.0$ , which weighed 134.66 lbs. Table 4 gives the design variables of the final optimal design which weighs 71.77 lbs. The constraints



that control the design are ring yield (.99), stringer yield (.99), skin yield (.99), and wall buckling (.98). The general buckling load is 263.88 lbs/in with mode number 7. Table 4 shows that the rings are minimum gage constrained in all segments and the wall thickness is approaching minimum gage (.016) in segment 6. The significant weight difference between this case and Case I clearly shows the effect of differing load application. Convergence to the optimal 71.77 lb design required 350 function evaluations.

Case V. This is a segment of a spherical shell as shown in Fig. 1. The shell is stiffened with exterior Z stringers and interior Z rings. The material is aluminum with  $\sigma_y = 40,000$  psi,  $E = 10 \times 10^6$  psi, and density  $\gamma = .1$  lbs/in<sup>3</sup>. The shell is loaded with a uniform external pressure of 10 psi. The stringers are continuous. The design process started with a uniform design with  $t_w = .6$ ,  $D_s = .75$ ,  $D_R = .75$ ,  $S_s = 2.0$ ,  $S_R = 2.0$ , which weighed 380.4 lbs. Table 5 gives the design variables of the final optimal design which weighs 289.91 lbs. The constraints that control the design are ring yield (.89), skin yield (.99), and wall buckling (1.0). The general buckling load is 22.75 psi and the critical mode is 19. No minimum gage constraints were active. Convergence to the final design required 285 function evaluations.

### Discussion

The 5 cases presented above demonstrate some of the versatility of the shell design scheme presented in this paper. Actually the scheme is much more versatile than the results presented here indicate. Discrete rings can be included in the process. Their properties are prescribed parameters which enter into the matrices  $K$ ,  $k$ , and  $\kappa$ . Also, a different shell wall geometry can be considered in each segment, i.e., stringers could be included in segments 1 and 6 but not in the other segments if this is appropriate. Also, multiple load conditions can be included. The computer program which was used to obtain the results in the preceding section of this paper has this versatility built into it.

The results presented also both indicate and mask some of the underlying difficulties which are encountered in numerical optimum design. An

indication of difficulty is the fact that the weight of Case III was 7% above the weight of Case II when it was expected that Case III would weigh less. The difficulty here probably lies in convergence criteria, starting point, just when the general buckling constraint is included in the process, how fast  $\rho$  is reduced, and the starting value of  $\rho$ . All of these factors are at the disposal of the designer and the quality of the results obtained are directly proportional to his skill in manipulating these parameters. This aspect of numerical optimum design is well known to its practitioners but is usually not apparent to the uninitiated. The design method presented here and all others known to the authors' require about as much art in their use as there is science in their development.

### Conclusions

The shell design problem is formulated and solved using first order differential equations and forward integration techniques. All derivatives are obtained exactly. Because of the self adjoint nature of the prebuckling and buckling problems, these derivatives can be obtained with a minimum of calculation (one additional particular solution to the prebuckling problem and the evaluation of an integral over the shell meridian (Eq. 19) for each design variable. Either 25 or 30 design variables are considered and convergence to the final designs are obtained in 300-450 function evaluations. This many function evaluations for a problem involving 25 to 30 design variables is well within the number of function evaluations which can be expected to be required and thus demonstrates the efficiency of the design process. The results indicate that, as with all other numerical optimum design techniques, care must be taken in the choice of convergence criteria and other parameters (e.g.,  $\rho$ ) which control the design process.

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Table 1. Case I 70° Mg Cone with Exterior Z Stringers, Interior O Rings, Weighing 51.52 lbs.

Shell Segment	Wall Thickness	Stringer Thickness	Ring Thickness	Stringer Depth	Ring Diameter	Stringer Spacing	Ring Spacing
	$t_w$	$t_s^a$	$t_R^b$	$D_S$	$D_R$	$S_S^d$	
	inches	inches	inches	inches	inches	inches	inches
1	.041	.023	.01 <sup>c</sup>	.79	.74	3.00	1.93
2	.042	.023	.01 <sup>c</sup>	.81	.96	3.00	1.80
3	.037	.027	.01 <sup>c</sup>	.95	1.15	3.00	1.68
4	.062	.031	.01 <sup>c</sup>	1.09	.93	3.00	1.65
5	.028	.033	.01 <sup>c</sup>	1.16	.74	3.00	1.68
6	.018	.033	.01 <sup>c</sup>	1.15	.57	3.00	2.03

<sup>a</sup>  $t_s = \max (\text{Eq. 12}, .01)$

<sup>b</sup>  $t_R = \max(\text{Eq. 10}, .01)$

<sup>c</sup> Minimum gage.

<sup>d</sup> Stringer spacing at small end of cone (continuous stringers).

Table 2. Case II 70° Mg Cone with Interior Z Stringers and Rings Weighing 53.83.

Shell Segment	Wall Thickness $t_w$ inches	Stringer Thickness $t_s^a$ inches	Ring Thickness $t_R^b$ inches	Stringer Depth $D_s$ inches	Ring Diameter $D_R$ inches	Stringer Spacing $s_s^c$ inches	Ring Spacing $s_R$ inches
1	.019	.028	.020	.97	.70	1.62	1.27
2	.021	.028	.019	1.00	.68	1.62	1.26
3	.028	.030	.019	1.04	.69	1.62	1.23
4	.029	.031	.019	1.07	.69	1.62	1.22
5	.029	.031	.018	1.08	.62	1.62	1.23
6	.026	.029	.016	1.03	.55	1.62	1.23

<sup>a</sup>  $t_s = \max (\text{Eq. 12}, .01).$

<sup>b</sup>  $t_R = \max (\text{Eq. 12}, .01).$

<sup>c</sup> Stringer spacing at small end of cone (continuous stringers).

Table 3. Case III 70° Mg Cone with Interior Z Stringers and Rings, Discontinuous Stringers, Weighing 57.58 lbs.

Shell Segment	Wall Thickness $t_w$ inches	Stringer Thickness $t_s^a$ inches	Ring Thickness $t_R^b$ inches	Stringer Depth $D_S$ inches	Ring Diameter $D_R$ inches	Stringer Spacing $S_S^c$ inches	Ring Spacing $S_R$ inches
1	.025	.025	.018	.86	.64	1.48	1.36
2	.026	.024	.019	.83	.67	1.49	1.35
3	.034	.024	.021	.82	.73	1.48	1.30
4	.036	.024	.020	.83	.71	1.52	1.30
5	.036	.023	.018	.81	.63	1.52	1.31
6	.029	.021	.017	.74	.60	1.53	1.34

<sup>a</sup>  $t_s = \max (\text{Eq. 12, .01}).$

<sup>b</sup>  $t_R = \max (\text{Eq. 12, .01}).$

<sup>c</sup> Stringer spacing at the small end of each segment.

Table 4. Case IV 70° Mg Cone with Exterior Z Stringers, Interior O Rings, Weighing 71.77 lbs.

Shell Segment	Wall Thickness $t_w$ inches	Stringer Thickness $t_S^a$ inches	Ring Thickness $t_R^b$ inches	Stringer Depth $D_S$ inches	Ring Diameter $D_R$ inches	Stringer Spacing $S_S^d$ inches	Ring Spacing $S_R$ inches
1	.035	.038	.01 <sup>c</sup>	1.32	1.21	2.64	2.22
2	.055	.042	.01 <sup>c</sup>	1.46	1.10	2.64	2.19
3	.060	.059	.01 <sup>c</sup>	2.05	1.03	2.64	2.19
4	.041	.059	.01 <sup>c</sup>	2.07	.87	2.64	2.21
5	.032	.047	.01 <sup>c</sup>	1.63	.85	2.64	2.20
6	.019	.039	.01 <sup>c</sup>	1.37	.77	2.64	2.23

<sup>a</sup>  $t_S = \max (\text{Eq. 12}, .01).$

<sup>b</sup>  $t_R = \max (\text{Eq. 10}, .01).$

<sup>c</sup> Minimum gage.

<sup>d</sup> Stringer spacing at small end of cone.



Table 5. Case VI A1 Sphere Segment with Exterior Z Stringers and Interior Z Rings, Weighing 289.91 lbs.

Shell Segment	Wall Thickness $t_w$ inches	Stringer Thickness $t_S^a$ inches	Ring Thickness $t_R^b$ inches	Stringer Depth $D_S$ inches	Ring Diameter $D_R$ inches	Stringer Spacing $S_S^c$ inches	Ring Spacing $S_R$ inches
1	.048	.024	.024	.71	.70	2.07	2.01
2	.046	.024	.023	.71	.68	2.07	2.01
3	.043	.025	.023	.71	.66	2.07	2.01
4	.043	.024	.023	.72	.66	2.07	2.01
5	.044	.023	.023	.66	.66	2.07	2.01
6	.044	.026	.022	.75	.64	2.07	2.00

<sup>a</sup>  $t_S = \max (\text{Eq. 12}, .01).$

<sup>b</sup>  $t_R = \max (\text{Eq. 17}, .01).$

<sup>c</sup> Stringer spacing at small end of spherical segment.

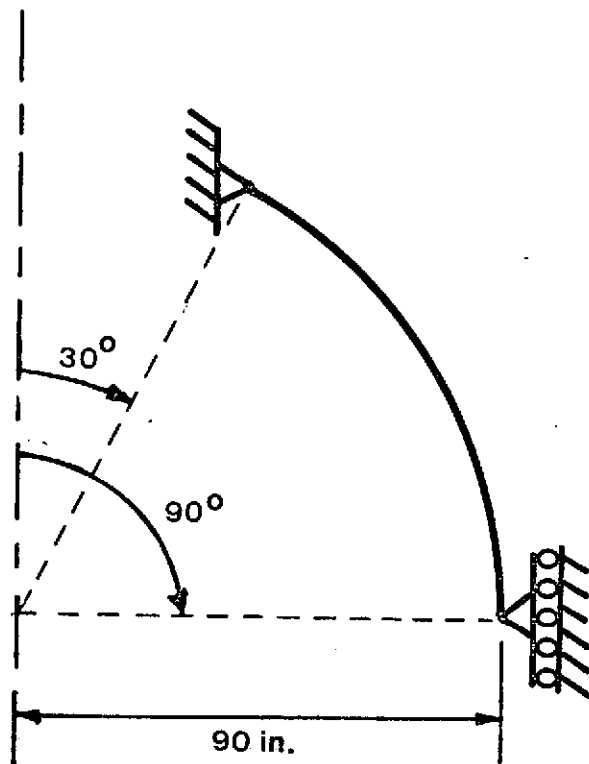


Fig. 1. Geometry and boundary conditions for spherical segment, Case V.

Figure Caption

Fig. 1. Geometry and boundary condition for spherical segment,  
Case V.